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Erratum

Erratum to "Inverse nodal problem for differential operator with eigenvalue in the boundary condition" [Appl. Math. Lett. 21 (2008) 1301–1305]

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ABSTRACT

In this paper, some mistakes in the paper which is cited in the title are corrected. Fortunately, the mistakes do not affect the validity of the main results.

1. Introduction

Consider the Sturm–Liouville problem

$$y'' + \left[\lambda^2 + \mu - q(x)\right] y = 0 \tag{1.1}$$

with the boundary conditions

$$y(0) = 0, \tag{1.2}$$

$$ay'(\pi, \lambda) + \lambda y(\pi, \lambda) = 0, \tag{1.3}$$

where $q(x)$ is real and integrable on $[0, \pi]$, μ and $a \neq 0$ are real parameters.

Let $\lambda_o(q, a) < \lambda_1(q, a) < \cdots \to \infty$ be the eigenvalues of (1.1)–(1.3) and $0 < x_1^n < \cdots < x_j^n < \pi, j = 1, 2, \ldots n-1$, be nodal points of the nth eigenfunction.

We denote

$$X(q, a) = \left\{x_j^n | j = 1, 2, \ldots n-1, n = 2, 3, \ldots\right\}$$

and the nodal space as

$$NS[0, \pi] = \left\{X(q, a) | a \in R, q \in L^1[0, \pi]\right\}.$$

In Hochstadt's paper [2], it is sought a solution of (1.1) in the form

$$y = \sum_{k=0}^{\infty} y_k,$$

$$y_0 = \sin \lambda x,$$

$$y_{k+1} = \int_0^x \frac{\sin \lambda (x-t)}{\lambda} [q(t) - \mu] y_k(t).$$

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By this induction, we see that

$$y = \sin \lambda x + \int_0^x \frac{\sin \lambda (x-t)}{\lambda} (q-\mu) \sin \lambda t \mathrm{d}t + o\left(\frac{1}{\lambda^2}\right), \tag{1.4}$$

$$y' = \lambda \cos \lambda x + \int_0^x \cos \lambda (x-t) (q-\mu) \sin \lambda t \mathrm{d}t + o\left(\frac{1}{\lambda}\right) \tag{1.5}$$

so that

$$\omega(\lambda_n) = ay'(\pi, \lambda) + \lambda y(\pi, \lambda) = \lambda (a \cos \lambda \pi + \sin \lambda \pi) + \frac{a \sin \lambda \pi - \cos \lambda \pi}{2} \int_0^\pi (q-\mu) \, \mathrm{d}t + o(1).$$

From the above we get [2]

$$\lambda_n = n + \lambda_o + \frac{1}{2\pi n} \int_0^\pi (q-\mu) \, \mathrm{d}t + o\left(\frac{1}{n}\right), \tag{1.6}$$

where $\tan \lambda_o \pi = -a$ and $|\lambda_o \pi| < \frac{\pi}{2}$.

2. Main results

In this section, we give some uniqueness theorems and their proofs. We mention that these results are given for the classical Sturm–Liouville problem by Yang [1].

Lemma 2.1. *The following inequality is valid*

$$\left(j - \frac{1}{2}\right)\pi < \lambda_n x_j^n < \left(j + \frac{1}{2}\right)\pi. \tag{2.1}$$

Proof. From (1.4), we know

$$y\left(\frac{\left(j-\frac{1}{2}\right)\pi}{\lambda_n}\right) = \sin\left(j - \frac{1}{2}\right)\pi + o\left(\frac{1}{n}\right) = (-1)^{j+1},$$

$$y\left(\frac{\left(j-\frac{3}{2}\right)\pi}{\lambda_n}\right) = \sin\left(j + \frac{1}{2}\right)\pi + o\left(\frac{1}{n}\right) = (-1)^{j}.$$

Hence, $y\left(\left(j-\frac{1}{2}\right)\pi/\lambda_n\right)$ and $y\left(\left(j+\frac{1}{2}\right)\pi/\lambda_n\right)$ have different signs. There exists at least one zero of y on the interval $\left(\left(j+\frac{1}{2}\right)\pi/\lambda_n, \left(j-\frac{1}{2}\right)\pi/\lambda_n\right)$ for large n. On the other hand, y has $n-1$ zeros in $(0, \pi)$. This implies (2.1). ■

Lemma 2.2. *Let $x_j^n = x_j^n(q, a)$. Then*

$$\lambda_n x_j^n = j\pi + \frac{1}{2\lambda_n} \int_0^{x_j^n} [q(t) - \mu] \, \mathrm{d}t + o\left(\frac{1}{n}\right). \tag{2.2}$$

Proof. Since x_j^n are zeros of y,

$$y(x_j^n) = \sin \lambda_n x_j^n + \int_0^{x_j^n} \frac{\sin \lambda_n \left(x_j^n - t\right)}{\lambda_n} [q(t) - \mu] \sin \lambda_n t \mathrm{d}t + o\left(\frac{1}{\lambda_n^2}\right) = 0,$$

$$\sin \lambda_n x_j^n + \frac{\sin \lambda_n x_j^n}{\lambda_n} \int_0^{x_j^n} \cos \lambda_n t \, [q(t) - \mu] \sin \lambda_n t \mathrm{d}t - \frac{\cos \lambda_n x_j^n}{\lambda_n} \int_0^{x_j^n} \sin^2 \lambda_n t \, [q(t) - \mu] \, \mathrm{d}t + o\left(\frac{1}{\lambda_n^2}\right) = 0,$$

$$\tan \lambda_n x_j^n = \frac{1}{\lambda_n} \int_0^{x_j^n} \frac{1 - \cos 2\lambda_n t}{2} [q(t) - \mu] \sin \lambda_n t \mathrm{d}t - \frac{\tan \lambda_n x_j^n}{\lambda_n} \int_0^{x_j^n} \cos \lambda_n t \, [q(t) - \mu] \sin \lambda_n t \mathrm{d}t + o\left(\frac{1}{\lambda_n^2}\right).$$

From Lemma 2.1 and the Taylor expansion for the Tan function, it is obtained that

$$\lambda_n x_j^n = j\pi + \frac{1}{2\lambda_n} \int_0^{x_j^n} [q(t) - \mu] \, \mathrm{d}t + o\left(\frac{1}{n}\right). \quad ■$$

Lemma 2.3 (*[1]*).

$$\lim_{j/n\to x, j/n\neq x} n = \infty. \tag{2.3}$$

Theorem 2.1. *The potential* $q - \frac{1}{\pi}\int_0^\pi q(y)\mathrm{d}y$ *and* a *are uniquely determined by any dense subset of the nodes in* $[0, \pi]$.

Proof. We assume that $a, \tilde{a}$ are real numbers and $q, \tilde{q} \in L^1[0, \pi]$. Let

$$A = \left\{ \frac{j_k}{n_k} \middle| k = 1, 2, \ldots \right\} \subset \left\{ \frac{j}{n} \middle| j = 1, 2, \ldots n-1, n = 2, 3, \ldots \right\}$$

that $\overline{A} = [0, \pi]$ and for $\frac{j}{n} \in A$, $x_j^n(q, a) = \tilde{x}_j^n(\tilde{q}, \tilde{a})$. We use the symbols $\tilde{\lambda}, \tilde{y}$ and so on when we replace q, a by $\tilde{q}, \tilde{a}$. From (1.1), we have

$$\left[y'(x)\tilde{y}(x) - \tilde{y}'(x)y(x)\right]' = \left(q - \tilde{q} + \lambda_n^2 - \tilde{\lambda}_n^2\right) y\tilde{y}. \tag{2.4}$$

Integrating (2.4) from x_j^n to π for $j/n \to 0$, we obtain

$$\left(\frac{\lambda}{a} - \frac{\tilde{\lambda}}{\tilde{a}}\right) y_n(\pi)\tilde{y}_n(\pi) = \int_{x_j^n}^{\pi} \left(q - \tilde{q} - \lambda_n^2 + \tilde{\lambda}_n^2\right) y\tilde{y}\mathrm{d}x.$$

From the asymptotic forms of y and $\tilde{y}$, we can obtain that $\left|y(\pi, \lambda_n)\tilde{y}(\pi, \lambda_n)\right|$ is bounded away from zero. Finally, if we take a sequence x_j^n accumulating at π, the right-hand side tends to zero. Then, $a = \tilde{a}$. Integrating (2.4) from 0 to x_j^n $(j/n \in A)$, we obtain

$$\int_0^{x_j^n} \left[q - \tilde{q} + \frac{1}{\pi}\int_0^\pi \tilde{q}(x)\mathrm{d}x - \frac{1}{\pi}\int_0^\pi q(x)\mathrm{d}x + o(1)\right] \sin^2 nx\mathrm{d}x = 0.$$

Hence, from the Riemann–Lebesgue theorem, we yield

$$\int_0^x \left[q - \tilde{q} + \frac{1}{\pi}\int_0^\pi \tilde{q}(x)\mathrm{d}x - \frac{1}{\pi}\int_0^\pi q(x)\mathrm{d}x\right] \mathrm{d}x = \int_0^x \left[q - \tilde{q}\right] \cos 2nx\mathrm{d}x = 0.$$

Therefore,

$$q - \frac{1}{\pi}\int_0^\pi q(x)\mathrm{d}x = \tilde{q} - \frac{1}{\pi}\int_0^\pi \tilde{q}(x)\mathrm{d}x \quad \text{almost everywhere on } [0, \pi]. \quad \blacksquare$$

Lemma 2.4.

$$\lim_{j/n\to x, j/n\neq x} \left((n + \lambda_0)\, x_j^n - j\pi\right) n = \frac{1}{2}\int_0^x [q(t) - \mu]\,\mathrm{d}t - x\left[\frac{1}{2}\int_0^\pi [q(t) - \mu]\,\mathrm{d}t\right].$$

Proof. From Lemma 2.3 we know

$$\lim_{j/n\to x, j/n\neq x} n = \infty. \tag{2.5}$$

From (1.6)

$$\frac{\lambda_n}{n} = 1 + \frac{\lambda_0}{n} + \frac{1}{2\pi n^2}\int_0^\pi [q(t) - \mu]\,\mathrm{d}t + O\left(\frac{1}{n^2}\right),$$

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = 1, \tag{2.6}$$

and

$$(\lambda_n - \lambda_0 - n)\, n = \frac{1}{2\pi}\int_0^\pi [q(t) - \mu]\,\mathrm{d}t + O\left(\frac{1}{n}\right).$$

Then,

$$\lim_{j/n\to x, j/n\neq x} (\lambda_n - \lambda_0 - n)\, n = \frac{1}{2\pi}\int_0^\pi [q(t) - \mu]\,\mathrm{d}t. \tag{2.7}$$

Combining the limits (2.5), (2.6) and Lemma 2.2, we have

$$\lim_{j/n\to x, j/n\neq x} x_j^n = x. \tag{2.8}$$

Again from Lemma 2.2,

$$\lim_{j/n\to x, j/n\neq x} \left(\lambda_n x_j^n - j\pi\right) n = \frac{1}{2\pi}\int_0^x [q(t)-\mu]\,\mathrm{d}t. \tag{2.9}$$

Then, considering (2.9) with (2.7), we finally obtain

$$\begin{aligned}\lim_{j/n\to x, j/n\neq x} \left((n+\lambda_0)\, x_j^n - j\pi\right) n &= \lim_{j/n\to x, j/n\neq x} \left(\lambda_n x_j^n - j\pi + (n+\lambda_0-\lambda_n)\, x_j^n\right) n \\ &= \frac{1}{2}\int_0^x [q(t)-\mu]\,\mathrm{d}t - x\left[\frac{1}{2}\int_0^\pi [q(t)-\mu]\,\mathrm{d}t\right]. \quad \blacksquare\end{aligned}$$

Theorem 2.2. *Let* $X = \left\{x_j^n | j = 1, 2, \ldots, n-1;\ n = 2, 3, \ldots\right\} \in NS\,[0, \pi]$. *Then,*

(I) *The following limit exists*

$$\lim_{j/n\to x, j/n\neq x} \left((n+\lambda_0)\, x_j^n - j\pi\right) n, \tag{2.10}$$

for all $x \in [0, \pi]$. *We denote the limit* $g(x)$, $x \in [0, \pi]$. *Then* $g(x)$ *is absolutely continuous and* $\frac{\mathrm{d}g(x)}{\mathrm{d}x}$ *exists almost everywhere.*

(II) *If we take*

$$q(x) = 2\frac{\mathrm{d}g(x)}{\mathrm{d}x},$$

then $X(q, a) = X$ *and* $\int_0^\pi q(t)\mathrm{d}t = 0$.

Proof. For any $X \in NS\,[0, \pi]$, from the definition there exists $q \in L^1\,[0, \pi]$ such that $X = X\,(q, a)$. By using Lemma 2.4, we know that the limit (2.10) exists. Again from Lemma 2.4, we know that

$$g(x) = \frac{1}{2\pi}\int_0^x [q(t)-\mu]\,\mathrm{d}t - x\left[\frac{1}{2\pi}\int_0^\pi [q(t)-\mu]\,\mathrm{d}t\right].$$

If we derive both sides of the last equation, obtaining

$$g^{'}(x) = \frac{1}{2}\left[q(x) - \frac{1}{2\pi}\int_0^\pi [q(t)]\,\mathrm{d}t\right] \quad \text{on } [0, \pi],$$

then

$$\int_0^\pi [q(t)]\,\mathrm{d}t = 0.$$

This completes the proof. $\blacksquare$

References

[1] X.F. Yang, A solution of the inverse nodal problem, Inverse Problems 13 (1997) 203–213.

[2] H. Hochstadt, On inverse problems associated with second-order differential operators, Acta Math. 119 (1967) 173–192.